Lesson 26: Volume and Surface Area

Student Outcomes

- Students solve real-world and mathematical problems involving volume and surface areas of three-dimensional objects composed of cubes and right prisms.

Related Topics: More Lesson Plans for Grade 7 Common Core Math

Lesson Notes

In this lesson, students apply what they learned in Lessons 22–25 to solve real-world problems. As students work the problems, encourage them to present their approaches for determining volume and surface area. Students use volume formulas to find the volume of a right prism that has had part of its volume displaced by another prism. Students work with cubic units and units of liquid measure on the volume problems. Students also will continue to calculate surface area.

Classwork

Opening (2 minutes)

In the Opening Exercise, students are asked to find the area of a region obtained by cutting a smaller rectangle out of the middle of a larger rectangle. This exercise provides information about students who may need some additional support during the lesson if they have difficulty solving this problem. Tell the class that today they will be applying what they learned about finding the surface area and volume of prisms to real-world problems.

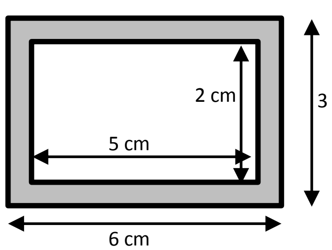
Opening Exercise (3 minutes)

Scaffolding:

If students are struggling, have them actually cut out the figures and take actual measurements.

Opening Exercise

Explain to your partner how you would calculate the area of the shaded region. Then, calculate the area.



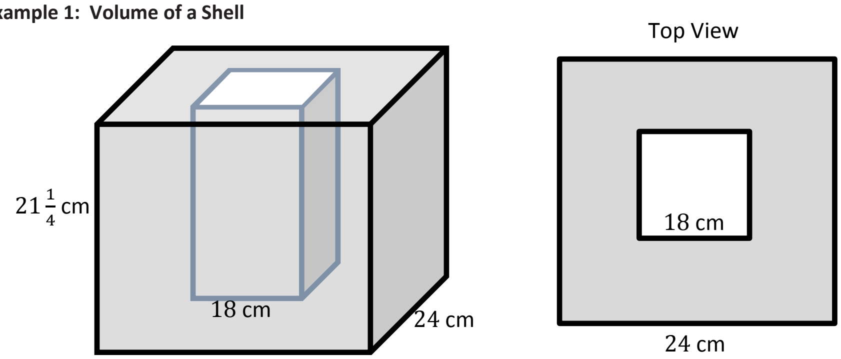
Find the area of the outer rectangle and subtract the area of the inner rectangle.

$$6\ cm \times 3\ cm - 5\ cm \times 2\ cm = 8\ cm^2$$

Example 1 (6 minutes): Volume of a Shell

This example builds on the area problem in the Opening Exercise, but extends to volume by cutting a smaller rectangular prism out of a larger cube to form an insulated box. Depending on the level of your students, you can guide them through this example, allow them to work with a partner, or allow them to work in small groups. If you have students work with a partner or a group, be sure to present different solutions and to monitor the groups' progress.

Example 1: Volume of a Shell



The insulated box shown is made from a large cube with a hollow inside that is a right rectangular prism with a square base. The figure at right is what the box looks like from above.

a. Calculate the volume of the outer box.

$24\ cm \times 24\ cm \times 24\ cm = 13,824\ cm^3$

b. Calculate the volume of the inner prism.

$18\ cm \times 18\ cm \times 21\frac{1}{4}\ cm = 6885\ cm^3$

c. Describe in words how you would find the volume of the insulation.

Find the volume of the outer cube; then, subtract the volume of the inner right rectangular prism.

d. Calculate the volume of the insulation in cubic centimeters.

$13,824\ cm^3 - 6885\ cm^3 = 6939\ cm^3$

e. Calculate the amount of water the box can hold in liters.

$6939\ cm^3 = 6939\ mL = \dfrac{(6939\ mL)}{1000\frac{mL}{L}} = 6.939\ L$

Use these questions with the whole class or small groups as discussion points.

- How did you calculate the volume of the insulation?
  - *First, calculate the volume of the outer cube, and then subtract the volume of the inner prism.*
- How do you convert cubic centimeters to liters?
  - $1\ cm^3 = 1\ mL$ *and* $1000\ mL = 1\ L$*, so divide by* 1000.

Exercise 1: Brick Planter Design (15 minutes)

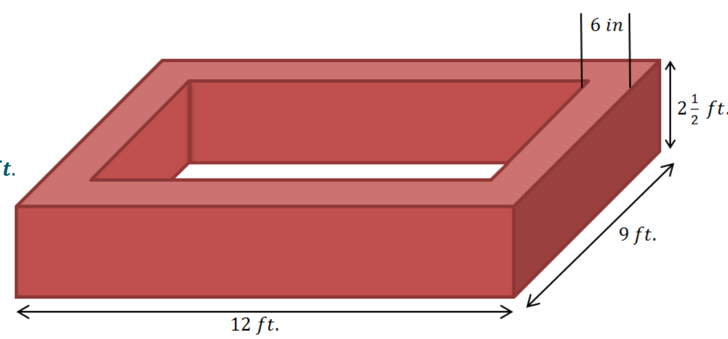
In this exercise, students will construct a brick planter and determine the amount of solid the planter will hold. First, students will calculate the number of bricks needed to build the planter; then, they will determine the cost of building the planter and filling it with soil. Have the class consider these questions as you discuss this exercise.

- How do you determine the internal dimensions?
  - *Find the external dimensions and subtract the thickness of the shell.*
- Explain how to determine the number of bricks needed. If you were going to construct this planter, can you think of other factors that would have to be considered?
  - *Find the volume of the bricks and divide by the volume of one brick. Other factors to consider include whether the bricks are perfectly rectangular and whether or not grout is used.*
- What do you think about your calculated cost? Is it what you expected? If not, is it higher or lower than you expected? Why?
  - *This is a very open-ended question and answers will depend on what the students originally thought. Some will say the cost is higher and others lower.*

Exercise 1: Designing a Brick Planter

You have been asked by your school to design a brick planter that will be used by classes to plant flowers. The planter will be built in the shape of a right rectangular prism with no bottom so water and roots can access the ground beneath. The exterior dimensions are to be 12 ft. $\times$ 9 ft. $\times$ $2\frac{1}{2}$ ft. The bricks used to construct the planter have are 6 inches long, $3\frac{1}{2}$ inches wide, and 2 inches high.

a. **What are the interior dimensions of the planter if the thickness of the planter's walls is equal to the length of the bricks?**



$6\ in = \frac{1}{2}\ ft.$

Interior length:

$12\ ft. - \frac{1}{2}\ ft. - \frac{1}{2}\ ft = 11\ ft.$

Interior width:

$9\ ft. - \frac{1}{2}\ ft. - \frac{1}{2}\ ft. = 8\ ft.$

Interior dimensions:

$11\ ft. \times 8\ ft. \times 2\frac{1}{2}\ ft.$

b. **What is the volume all the brick that forms the planter?**

Solution 1

Subtract the volume of the smaller interior prism V_S from the volume of the large exterior prism V_L.

$V_{Brick} = V_L - V_S$

$V_{Brick} = \left(12\ ft. \times 9\ ft. \times 2\frac{1}{2}\ ft.\right) - \left(11\ ft. \times 8\ ft. \times 2\frac{1}{2}\ ft.\right)$

$V_{Brick} = 270\ ft^3 - 220\ ft^3$

$V_{Brick} = 50\ ft^3$

Scaffolding:

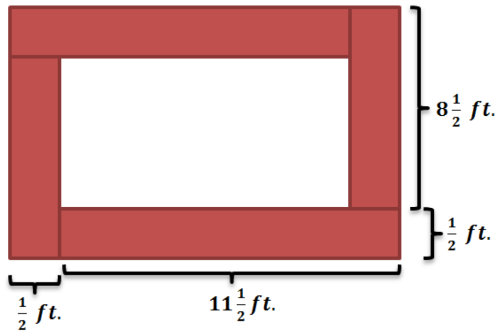
Solution 2 is an extension of thinking from Lesson 3 Example 6 in which students found various ways to write expressions representing a tiled perimeter around a rectangle.

Scaffolding:

If you have students that need a challenge, have them extend this concept to the volume of a cylinder and the volume of the metal part of a can.

Solution 2

The volume of the brick is equal to the area of the base times the height.



$B = \frac{1}{2}\ ft. \times \left(8\frac{1}{2}\ ft. + 11\frac{1}{2}\ ft. + 8\frac{1}{2}\ ft. + 11\frac{1}{2}\ ft.\right)$

$B = \frac{1}{2}\ ft. \times (40\ ft.) = 20\ ft^2$

$V = Bh$

$V = (20\ ft^2)\left(2\frac{1}{2}\ ft.\right) = 50\ ft^3$

c. **If you are going to fill the planter $\frac{3}{4}$ full of soil, how much soil will you need to purchase, and what will be the height of the soil?**

The height of the soil will be $\frac{3}{4}$ of $2\frac{1}{2}$ feet.

$\frac{3}{4}\left(\frac{5}{2}\ ft.\right) = \frac{15}{8}\ ft.$; *The height of the soil will be $\frac{15}{8}$ feet (or $1\frac{7}{8}$ feet).*

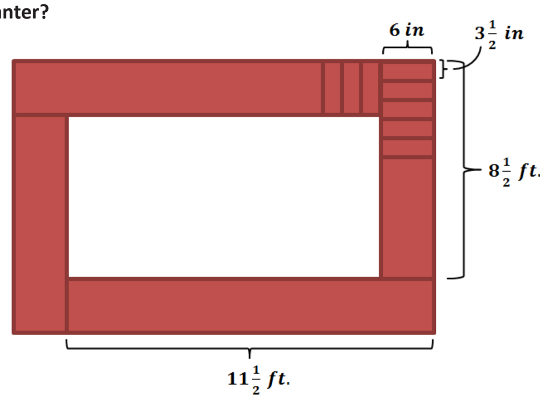
The volume of the soil in the planter:

$Volume = \left(11\ ft. \times 8\ ft. \times \frac{15}{8}\ ft.\right)$

$Volume = (11\ ft. \times 15\ ft.) = 165\ ft^3$

d. **How many bricks are needed to construct the planter?**

Solution 1:



$Perimeter = 2\left(8\frac{1}{2} ft.\right) + 2\left(11\frac{1}{2} ft.\right)$

$Perimeter = 17\ ft. + 23\ ft. = 40\ ft.$

$3\frac{1}{2}\ in = \frac{7}{24}\ ft$

The perimeter P is equal to the product of the number of bricks (in one layer) n times the width of a brick w.

$P = n \cdot w$

$40\ ft. = n\left(\frac{7}{24}\ ft\right)$

$\left(40\left(\frac{24}{7}\right)\right) = n$

$\frac{960}{7} \approx 137.1\ bricks$ *Each layer of the planter requires approximately 137.1 bricks.*

$2\ in = \frac{1}{6}\ ft$

The height of the planter $2\frac{1}{2}\ ft.$ *is equal to the product of the number of layers of brick* n *and the height of each brick* $\frac{1}{6}\ ft.$

$2\frac{1}{2}\ ft. = n\left(\frac{1}{6}\ ft.\right)$

$6\left(2\frac{1}{2}\right) = n$

$15 = n$; *There are* 15 *layers of bricks in the planter.*

The total number of bricks b *is equal to the product of the number of bricks in each layer* $\left(\frac{960}{7}\right)$ *and the number of layers* (15).

$b = \left(\frac{960}{7}\right)(15)$

$b = \frac{14400}{7} \approx 2057.1$

It is not reasonable to purchase 0.1 *bricks; we must round up to the next whole brick, which is* 2058 *bricks.*

e. **The bricks used in this project cost $\$0.82$ each and weigh 4.5 lb. each. The supply company charges a delivery fee of $\$15$ per whole ton (2000 lb.) over 4000 pounds. How much will your school pay for the bricks (including delivery) to construct the planter?**

If the school purchases 2060 *bricks, the total weight of the bricks for the planter:*

$2060(4.5) = 9270\ lb.$

The number of whole tons over 4000 *pounds:*

$9270 - 4000 = 5270$

Since $1\ ton = 2000\ lb.$, *there are* 2 *whole tons* $(4000\ lb)$ *in* $5270\ lb.$

$Total\ cost = cost\ of\ bricks + cost\ of\ delivery$

$Total\ cost = 0.82(2060) + 2(15)$

$Total\ cost = 1689.20 + 30 = 1719.20$

The cost for bricks and delivery will be $\$1719.20$.

f. **A cubic foot of top soil weighs between 75 and 100 lbs. How much will the soil in the planter weigh?**

The volume of the soil in the planter is $165\ ft^3$. *The weight of the soil in the planter:*

Minimum weight:

$Minimum\ weight = 75(165)$

$Minimum\ weight = 12,375\ lb.$

Maximum weight:

$Maximum\ weight = 100(165)$

$Maximum\ weight = 16,500\ lb.$

The soil in the planter will weigh between $12,375\ lb$ *and* $16,500\ lb.$

g. **If the topsoil costs $\$0.88$ per each cubic foot, calculate the total cost of materials that will be used to construct the planter.**

The total cost of the top soil:

$Cost = 0.88(165) = 145.2$; *The cost of the top soil will be* $\$145.20$.

The total cost of materials for the brick planter project:

$Cost = (cost\ of\ bricks) + (cost\ of\ soil)$

$Cost = \$1719.20 + \$145.20 = \$1864.40$

The total cost of materials for the brick planter project will be $\$1864.40$.

Exercise 2: Design a Feeder (12 minutes)

This is a very open-ended task. If students have struggled with the first example and exercise, you may wish to move them directly to some of the problem set exercises. Students may at first struggle to determine a figure that will work, but refer back to some of the designs from earlier lessons. Right prisms with triangular bases or trapezoidal bases would work well. Encourage students to find reasonable dimensions and to be sure the volume is in the specified range. Students may try various approaches to this problem. Students may work with partners or in groups, but be sure to bring the class back together and have students present their designs and costs. Discuss the pros and cons of each design. You could even have a contest for the best design.

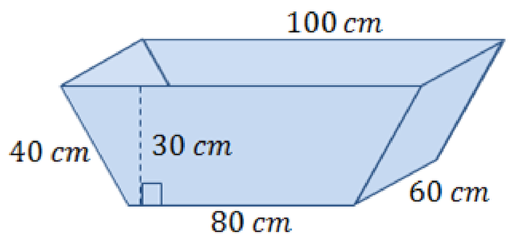
Exercise 2: Design a Feeder

You did such a good job designing the planter that a local farmer has asked you to design a feeder for the animals on his farm. Your feeder must be able to contain at least $100,000$ cubic centimeters, but not more than $200,000$ cubic centimeters of grain when it is full. The feeder is to be built of stainless steel and must be in the shape of a right prism, but not a right rectangular prism. Sketch your design below including dimensions. Calculate the volume of grain that it can hold and the amount of metal needed to construct the feeder.

The farmer needs a cost estimate. Calculate the cost of constructing the feeder if $\frac{1}{2}$ cm thick stainless steel sells for $\$93.25$ per square meter.

Answers will vary. Below is an example using a right trapezoidal prism.

This feeder design consists of an open-top container in the shape of a right trapezoidal prism. The trapezoidal sides of the feeder will the allow animals easier access to feed at its bottom. The dimensions of the feeder are shown in the diagram.



$B = \frac{1}{2}(b_1 + b_2)h$

$B = \frac{1}{2}(100\ cm + 80\ cm) \cdot 30\ cm$

$B = \frac{1}{2}(180\ cm) \cdot 30cm$

$B = 90\ cm \cdot 30\ cm = 2700\ cm^2$

$V = Bh$

$V = (2700\ cm^2)(60\ cm) = 162,000\ cm^3$;

The volume of the solid prism is $162,000\ cm^3$, so the volume that the feeder can contain is slightly less, depending on the thickness of the metal used.

The exterior surface area of the feeder tells us the area of metal required to build the feeder.

$SA = (LA - A_{top}) + 2B$

$SA = 60\ cm \cdot (40\ cm + 80\ cm + 40\ cm) + 2(2700\ cm^2)$

$SA = 60\ cm(160\ cm) + 5400\ cm^2$

$SA = 9600\ cm^2 + 5400\ cm^2 = 15000\ cm^2$

The feeder will require $15000\ cm^2$ of metal.

$1\ m^2 = 10000\ cm^2$, *so* $15000\ cm^2 = 1.5\ m^2$

$Cost = \$93.25(1.5\ m^2) = \$139.875$; ***Since this is a measure of money, the cost must be rounded to the nearest cent which is $\$139.88$.***

Closing (2 minutes)

- Describe the process of finding the volume of a prism shell.
  - *Find the volume of the outer figure; then, subtract the volume of the inner figure.*
- How does the thickness of the shell affect the internal dimensions of the prism? The internal volume? The external volume?
  - *The thicker the shell, the smaller the internal dimensions and the smaller the internal volume. The external volume is not affected by the thickness of the shell.*

Exit Ticket (5 minutes)

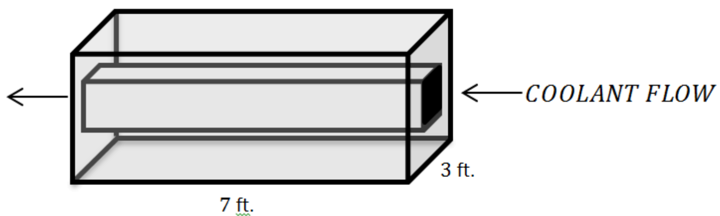
The Exit Ticket problem includes scaffolding. If students grasp this concept well, assign only parts (c) and (d) of the exit ticket.

Name ____ Date ____________________

Lesson 26: Volume and Surface Area

Exit Ticket

1. Lawrence is designing a cooling tank that is a square prism. A pipe in the shape of a smaller $2\ ft. \times 2\ ft.$ square prism passes through the center of the tank as shown in the diagram, through which a coolant will flow.



   a. What is the volume of the tank including the cooling pipe?

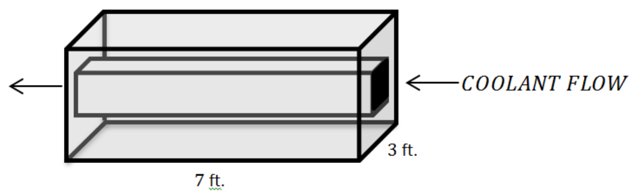
   b. What is the volume of coolant that fits inside the cooling pipe?

   c. What is the volume of the shell (the tank not including the cooling pipe)?

   d. Find the surface area of the cooling pipe.

Exit Ticket Sample Solutions

1. Lawrence is designing a cooling tank that is a square prism. A pipe in the shape of a smaller $2\ ft. \times 2\ ft.$ square prism passes through the center of the tank as shown in the diagram, through which a coolant will flow.



> *Scaffolding:*
> If students have mastered this concept easily, assign only parts (c) and (d).

a. What is the volume of the tank including the cooling pipe?

$7\ ft \times 3\ ft \times 3\ ft = 63\ ft^3$

b. What is the volume of coolant that fits inside the cooling pipe?

$2\ ft. \times 2\ ft. \times 7\ ft. = 28\ ft^3$

c. What is the volume of the shell (the tank not including the cooling pipe)?

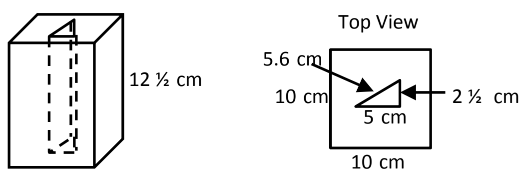
$63\ ft^3 - 28\ ft^3 = 35\ ft^3$

d. Find the surface area of the cooling pipe.

$2\ ft \times 7\ ft \times 4 = 56\ ft^2$

Problem Set Sample Solutions

1. A child's toy is constructed by cutting a right triangular prism out of a right rectangular prism.



a. Calculate the volume of the rectangular prism.

$10cm \times 10cm \times 12\frac{1}{2}cm = 1250\ cm^3$

b. Calculate the volume of the triangular prism.

$\frac{1}{2}\left(5\ cm \times 2\ \frac{1}{2}cm\right) \times 12\frac{1}{2}cm = 78\frac{1}{8}\ cm^3$

c. **Calculate the volume of the material remaining in the rectangular prism.**

$$1250\ cm^3 - 78\frac{1}{8}\ cm^3 = 1171\frac{7}{8}\ cm^3$$

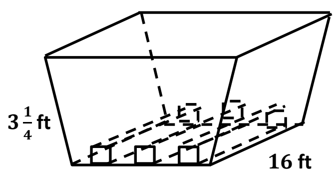
d. **What is the largest number of triangular prisms that can be cut from the rectangular prism?**

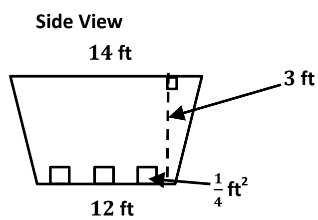
$$\frac{1250}{78\frac{1}{8}\ cm^3} = 16$$

e. **What is the surface area of the triangular prism (assume there is no top or bottom)?**

$$5.6\ cm \times 12\frac{1}{2} cm + 2\frac{1}{2} cm \times 12\frac{1}{2} cm + 5cm \times 12\ \frac{1}{2} cm = 163\frac{3}{4}\ cm^2$$

2. **A landscape designer is constructing a flower bed in the shape of a right trapezoidal prism. He needs to run three identical square prisms through the bed for drainage.**





a. **What is the volume of the bed without the drainage pipes?**

$$\frac{1}{2}(14\ ft + 12\ ft) \times 3\ ft \times 16\ ft = 624\ ft^3$$

b. **What is the total volume of the three drainage pipes?**

$$3\left(\frac{1}{4}\ ft^2 \times 16\ ft\right) = 12\ \ ft^3$$

c. **What is the volume of soil that can fit in the bed once the pipes are in place, assuming the amount of soil is filled to $\frac{3}{4}$ of the height of the planter?**

$$\frac{3}{4}(624\ ft^3) - 12\ ft^3 = 456\ ft^3 \ \ or\ \ \left[\frac{1}{2}(14\ ft + 12\ ft) \times \frac{3}{4}(3\ ft) \times 16\ ft\right] - 12\ ft^3 = 456\ ft^3$$

d. **What is the height of the soil?**

$$\frac{456\ ft^3}{\frac{1}{2}(14\ ft + 12\ ft) \times 16\ ft} = 2.2\ ft$$

e. If the bed is made of 8 ft $\times$ 4 ft pieces of plywood, how many pieces of plywood will the landscape designer need to construct the bed without the drainage pipes?

$$2\left(3\frac{1}{4}\ ft \times 16\ ft\right) + 12ft \times 16\ ft + 2\left(\frac{1}{2}(12\ ft + 14\ ft) \times 3\ ft\right) = 374\ ft^2$$

$$374\ ft^2 \div \frac{(8\ ft \times 4\ ft)}{piece\ of\ plywood} = 11.7\ or\ 12\ pieces\ of\ plywood$$

f. If the plywood needed to construct the bed costs $\$35$ per 8 ft. $\times$ 4 ft. piece, the drainage pipes cost $\$125$ each, and the soil costs $\$1.25$/cubic foot, how much does it cost to construct and fill the bed?

$$\frac{\$35}{piece\ of\ plywood}(12\ pieces\ of\ plywood) + \frac{\$125}{pipe}(3\ pipes) + \frac{\$1.25}{ft^3\ soil}(456\ ft^3\ soil) = \$1365.00$$